

Long-Range Dependence Does Not Necessarily Imply Non-Exponential Tails

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Abstract—We exhibit a buffer model with a provably long-range dependent input source but whose corresponding measure of buffer occupancy is exponentially distributed (and thus has exponential tails). This example invalidates the perception which has emerged in the literature on traffic engineering, that long-range dependencies necessarily induce nonexponential tails for the buffer asymptotics.

Index Terms—Exponential tails, long-range dependence, on-off fluid models.

I. INTRODUCTION

STARTING with the landmark data set collected at BellCore [13], a growing number of measurement studies have by now concluded that network traffic exhibits time dependencies at a much larger number of time scales than had been traditionally observed. This *long-range dependence* has been detected in a wide range of applications and over multiple networking infrastructures, e.g., Ethernet LANs [13] (and references therein), VBR traffic [3], [8], Web traffic [6], and WAN traffic [17].

Roughly speaking, this long-range dependence amounts to correlations in the packet stream spanning multiple time scales, which are individually rather small but which decay so slowly as to be nonsummable. This is expected to affect performance in a manner drastically different from that predicted by (traditional) summable correlation structures which typically arise in Markovian traffic models and Poisson-like sources. This “failure of Poisson modeling” has generated a strong interest in alternative traffic models which capture observed (long-range) dependencies [7], [14], e.g., fractional Brownian motion [15] and its discrete-time analog, fractional Gaussian noise [1], and on-off (fluid) sources with heavy-tailed and subexponential activity periods (e.g., [4] and [11]). These studies have exposed the limitations of traditional traffic models in predicting storage requirements and devising congestion controls, in that the buffer asymptotics found for these traffic inputs do not display the *exponential* tails typically associated with short-range dependent Markovian models.

In fact, in the wake of these and related studies, a perception has emerged in the literature to the effect that long-range dependencies necessarily induce *non-exponential* tails for the

buffer asymptotics. Here we strike a note of caution as to the validity of such a “folk” theorem. We do so by exhibiting a buffer model with a provably long-range dependent input source but whose corresponding measure of buffer occupancy is *exponentially* distributed (and thus has exponential tails). Specifically, we consider the popular on-off fluid model used for evaluating the performance of asynchronous transfer mode (ATM) multiplexers where an infinite capacity buffer fed by an (independent) on-off fluid source with peak rate r is drained at constant rate c .

As summarized in Section II, the statistics of such an on-off fluid source are fully determined by a pair of independent random variables (rvs) B and I describing the generic on-period and off-period durations, respectively. In Section III we easily adapt the results obtained in [10] to identify the polynomial decay of the correlation function of the on-off source with exponentially distributed B and heavy-tailed off-duration rv I (e.g., distributed according to a regularly varying or Pareto-like distribution). Such an on-off source is long-range dependent, in fact asymptotically (second-order) self-similar.

If $rp < c < r$ where p is the asymptotic fraction of time that the source is active, then there exists a nonidentically zero \mathbb{R}_+ -valued rv V , known as the stationary backlog, which characterizes buffer occupancy level in steady state. In Section IV a simple representation of the stationary backlog is given in terms of the stationary waiting time rv in an auxiliary stable $GI|GI|1$ queue. This representation is then used in Section V to outline a proof of the fact that if B is exponentially distributed, so is the rv $[V|V > 0]$ *regardless* of the distribution of I .

The discrete-time version of the model used here was already considered in [9]; it was pointed out in that reference that fitting the data sometimes requires that the off-period duration rv I be modeled by a heavy-tailed distribution, say a Pareto-like rv. This simple model crisply illustrates the complex and subtle impact that (long-range) dependencies in the input stream can have on the tail probability of buffer contents through the queue dynamics. In line with the discussion in [16], buffer sizing *cannot* be determined adequately by appealing solely to the short versus long-range dependence characterization, thus second-order properties, of the input traffic.

A word on the notation used in this letter. Two \mathbb{R} -valued rvs X and Y are said to be equal in law if they have the same distribution, a fact we denote by $X =_{st} Y$. Convergence in distribution is denoted by \Rightarrow_n (with n going to infinity). For any integrable \mathbb{R}_+ -valued rv X , the *forward recurrence* time X^* is defined as the rv with integrated tail distribution given by

$$\mathbf{P}[X^* > x] := \mathbf{E}[X]^{-1} \int_x^\infty \mathbf{P}[X > t] dt, \quad x \geq 0.$$

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II. ON-OFF SOURCES

An on-off source of peak rate r is characterized by a succession of cycles, each such cycle comprising an off-period followed by an on-period. During the on-periods the source is active and produces fluid at constant rate r ; the source is silent during the off-periods: For each $n = 0, 1, \dots$, let B_n and I_n denote the durations of the on-period and off-period in the $(n+1)$ th cycle, respectively. Thus, if the epochs $\{T_n, n = 0, 1, \dots\}$ denote the beginning of successive cycles, with $T_0 := 0$ we have $T_{n+1} := \sum_{\ell=0}^n I_\ell + B_\ell$ for each $n = 0, 1, \dots$. The activity of the source is then described by the $\{0,1\}$ -valued process $\{\xi(t), t \geq 0\}$ given by

$$\xi(t) := \sum_{n=0}^{\infty} \mathbf{1}[T_n + I_n \leq t < T_{n+1}], \quad t \geq 0 \quad (1)$$

with the source active (resp. silent) at time t if $\xi(t) = 1$ (resp. $\xi(t) = 0$).

An *independent on-off* source is one for which: 1) the \mathbb{R}_+ -valued rvs $\{I_n, n = 1, \dots\}$ and $\{B_n, n = 1, \dots\}$ are mutually independent rvs which are independent of the pair of rvs I_0 and B_0 associated with the initial cycle and 2) the rvs $\{I_n, n = 1, \dots\}$ (resp. $\{B_n, n = 1, \dots\}$) are i.i.d. rvs with generic off-period duration rv I (resp. on-period duration rv B). Throughout the generic rvs B and I are assumed to be independent \mathbb{R}_+ -valued rvs such that $0 < \mathbf{E}[B], \mathbf{E}[I] < \infty$, and we simply refer to the independent on-off process just defined as the on-off source (B, I) .

In general, the activity process (1) is not stationary unless the rvs I_0 and B_0 are selected appropriately. Here we use the following variation on a construction given in [2] and [18]. With

$$p := \mathbf{E}[B] (\mathbf{E}[B] + \mathbf{E}[I])^{-1}, \quad (2)$$

we introduce the $\{0,1\}$ -valued rv U distributed according to $\mathbf{P}[U = 1] = p = 1 - \mathbf{P}[U = 0]$. A stationary version of (1), still denoted $\{\xi(t), t \geq 0\}$, is now obtained by selecting (I_0, B_0) to be of the form

$$(I_0, B_0) =_{st} (0, B^*)U + (I^*, B)(1 - U) \quad (3)$$

with rvs U, B, B^* and I^* taken to be mutually independent and independent of the rvs $\{B_n, I_n, n = 1, \dots\}$.

III. CORRELATION STRUCTURE

If $\{\xi(t), t \geq 0\}$ is the stationary version of the on-off source (B, I) , then its correlation function is defined by

$$\Gamma(h) := \text{cov}(\xi(t), \xi(t+h)), \quad t, h \geq 0. \quad (4)$$

Below we shall write $\Gamma(h; B, I)$ to acknowledge the fact that the correlation function (4) is determined by the rvs B and I . An expression for this correlation function was obtained in [10, Th. 2.2, p. 148], and used there to derive large lag asymptotics [10, Th. 4.3, p. 158]. Before specializing these results to the case of interest, we recall that an \mathbb{R}_+ -valued rv X is said to be of regular variation if

$$\mathbf{P}[X > x] = x^{-\alpha} L(x), \quad x > 0 \quad (5)$$

for some constant $\alpha > 0$ and some slowly varying function $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The constant α is known as the index of

variation, and we write $X =_{st} RV_\alpha$. Moreover for any $\mu > 0$, we denote by E_μ any rv which is exponentially distributed with parameter μ .

We now specialize Theorem 4.3 in [10, p. 158] to the on-off source (B, I) with $B =_{st} RV_\alpha$ and $I =_{st} E_\beta$ for constants $1 < \alpha < 2$ and $\beta > 0$. In that case, we have $\mathbf{P}[I > t] = o(\mathbf{P}[B > t])$ (with t going to infinity). Under a nonsingularity assumption on the distribution of the sum $I_1 + B_1 + \dots + I_n + B_n$ for some $n = 1, 2, \dots$, [10, p. 158, eq. (4.11)] holds and takes the form

$$\Gamma(h; RV_\alpha, E_\beta) \sim \frac{(1-p)^3}{(\alpha-1)\mathbf{E}[I]} h^{-(\alpha-1)} L(h) \quad (6)$$

(as h goes to infinity). For instance, the required nonsingularity assumption is satisfied if I is of regular variation of the form

$$\mathbf{P}[I > x] = a^\alpha (a+x)^{-\alpha}, \quad x \geq 0 \quad (7)$$

for $1 < \alpha < 2$ and $a > 0$.

We now turn to the case of an on-off source (B, I) with $B =_{st} E_\beta$ and $I =_{st} RV_\alpha$ for constants $1 < \alpha < 2$ and $\beta > 0$. The key observation is that if $\{\xi(t), t \geq 0\}$ is the stationary version of the on-off source (B, I) , then $\{1 - \xi(t), t \geq 0\}$ can be interpreted as the stationary version of the on-off source (I, B) . Thus

$$\Gamma(h; B, I) = \Gamma(h; I, B), \quad h \geq 0 \quad (8)$$

and whenever (6) holds, we can also conclude that

$$\Gamma(h; E_\beta, RV_\alpha) \sim \frac{(1-p)^3}{(\alpha-1)\mathbf{E}[I]} h^{-(\alpha-1)} L(h) \quad (9)$$

(as h goes to infinity). Consequently, here as well, we obtain the asymptotic self-similarity, thus long-range dependence, of the on-off source (B, I) with $B =_{st} E_\beta$ and $I =_{st} RV_\alpha$ for constants $1 < \alpha < 2$ and $\beta > 0$. A similar result was obtained in [9] for discrete-time on-off sources by direct arguments.

IV. THE STATIONARY BACKLOG

Consider an independent on-off source (B, I) with peak rate r as described in Section II. The total amount $A(t)$ of fluid generated in $[0, t)$ by this on-off source is given by

$$A(t) = r \int_0^t \xi(s) ds, \quad t \geq 0. \quad (10)$$

If we offer this on-off source $\{A(t), t \geq 0\}$ to an infinite capacity buffer drained at the constant rate of c , then under the nontriviality condition $c < r$, a backlog results in the amount $V(t)$ at time $t \geq 0$. Under the stability condition

$$rp < c \quad (11)$$

with p given by (2), there exists a nonidentically zero \mathbb{R}_+ -valued rv V such that $V(t) \Rightarrow_t V$ irrespectively of the initial backlog $V(0)$. The rv V is known as the *stationary backlog* and can be represented by

$$V =_{st} \sup_{t \geq 0} (A(t) - ct) \quad (12)$$

where $\{A(t), t \geq 0\}$ is given by (10) evaluated with the stationary (and reversible) version of $\{\xi(t), t \geq 0\}$.

For our purpose we find it useful to relate the stationary backlog as given by (12) to the stationary waiting of an auxiliary $GI|GI|1$ queue. First some notation: Consider a standard $GI|GI|1$ queue with generic service time σ and interarrival time τ ; these rvs are assumed integrable. For each $n = 0, 1, \dots$, let W_n denote the waiting time (in buffer) of the n^{th} customer. Under the stability condition

$$\mathbf{E}[\sigma] < \mathbf{E}[\tau] \quad (13)$$

there exists an \mathbb{R}_+ -valued rv $W(\sigma, \tau)$ such that $W_n \Rightarrow_n W(\sigma, \tau)$ irrespectively of W_0 . We refer to $W(\sigma, \tau)$ as the *stationary waiting time* rv associated with the standard $GI|GI|1$ queue with generic service time σ and interarrival time τ .

The following result has appeared elsewhere [5], [12] in a somewhat different form for a model equivalent to the one considered here:

Proposition 1: Consider the buffer model with drain rate c when fed by an on-off source (B, I) with peak rate r such that $rp < c < r$, and define the rv X_0 by

$$X_0 =_{st} (r - c)B^*U + ((r - c)B - cI^*)(1 - U)$$

where the rvs B, B^*, I^* and U are mutually independent. Then, it holds that

$$V =_{st} \left[X_0 + W((r - c)B, cI) \right]^+$$

with the rv X_0 taken independent of the rv $W((r - c)B, cI)$. Note that (11) is equivalent to (13) with the identification $\sigma = (r - c)B$ and $\tau = cI$. Proposition 1 can be obtained by setting $h = 0$ in Proposition 4.1 in [2].

V. EXPONENTIAL ON-PERIODS

Closed-form expressions can be obtained for the distribution of V when the on-period durations are exponentially distributed, say $B =_{st} E_\beta$ for some parameter $\beta > 0$, so that

$$(r - c)B =_{st} E_\mu \quad \text{with} \quad \mu = (r - c)^{-1}\beta. \quad (14)$$

Noting that $B^* =_{st} B$, we conclude that

$$V =_{st} \left[\tilde{X}_0 + (r - c)B + W((r - c)B, cI) \right]^+ \quad (15)$$

with rvs \tilde{X}_0, B and $W((r - c)B, cI)$ mutually independent, and $\tilde{X}_0 =_{st} -cI^*(1 - U)$.

The key observation is that $W((r - c)B, cI) + (r - c)B$ can now be interpreted as the stationary delay $D((r - c)B, cI)$ in a stable $GI|M|1$ with generic service time $(r - c)B =_{st} E_\mu$ and interarrival time cI . It is well known that the stationary delay in a stable $GI|M|1$ queue is exponentially distributed [19, p. 395]: More precisely, $D((r - c)B, cI) =_{st} E_{\sigma(1-\mu)}$ where σ is the unique solution to the nonlinear equation

$$\sigma = \mathbf{E} \left[e^{-\mu(1-\sigma)cI} \right], \quad 0 < \sigma < 1. \quad (16)$$

Consequently, with independent rvs \tilde{X}_0 and $E_{\sigma(1-\mu)}$, we can rewrite (15) as

$$V =_{st} \left[\tilde{X}_0 + E_{\sigma(1-\mu)} \right]^+. \quad (17)$$

Straightforward computations lead to the following result which appeared already in [5], [12] in the context of a manufacturing model with random disruptions.

Proposition 2: Under the assumptions of Proposition 1, assume that $B =_{st} E_\beta$ for some parameter β . Then, with μ given through (14), it holds that

$$\mathbf{P}[V > t] = \frac{pr}{c} e^{-\mu(1-\sigma)t}, \quad t \geq 0. \quad (18)$$

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